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Response of Flight Vehicles to Nonstationary Atmospheric Turbulence

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Exploratory studies are made on the feasibility of using the theory of nonstationary random processes in the vehicle-response analysis. To this end, response statistics are computed for the plunging rigid body motion of the vehicle traveling in a nonstationary turbulence field. The approach is an adaptation of Priestley's evolutionary spectral analysis. Further assuming that the turbulence field is Gaussian distributed expressions for the ensemble average of the threshold crossing rate and the probability density of response peak magnitude conditional on the occurrence of a peak are derived. Numerical examples presented in the paper clearly demonstrate that a nonstationary analysis is not only feasible but it also rectifies an unconservative aspect of the traditional stationary approach which cannot account for possible transient overloads.

Introduction

THE statistical approach to the dynamic analysis of flight vehicles subjected to atmospheric turbulence usually presupposes that the turbulence field is homogeneous and isotropic.[1] Then, in the simplest idealization, where a flight vehicle is treated as a single-degree-of-freedom system, (considering only the plunging rigid body motion) the forcing function in the governing differential equation is a stationary random process. The mean square value of the steady-state system response can be obtained, therefore, from an integration over frequency of the product of the spectral density of the gust velocity and the squared absolute value of an appropriate frequency-response function. Many such analyses are available in the literature.[2] Two commonly used spectral densities for the gust velocity are the Dryden spectrum and the von Kármán spectrum. Also, the gust velocity is generally assumed to be a Gaussian random process for the convenience of computing the statistics of threshold crossing and peak magnitude of the response.

It has been verified experimentally[3,4] that the gust velocity is approximately Gaussian distributed.‡ However, strong nonstationarity characteristics are evident in some cases, especially, for low-altitude turbulence over rough terrain. That a nonstationary analysis is needed is further evidenced by the recent treatment of the mean square gust velocity as a random variable.[6]

The purpose of this paper is to explore the feasibility of modeling atmospheric turbulence by a nonstationary random process in the vehicle response analysis. Sample calculations will be presented on the statistics of the response including threshold crossings and peak distribution. In order that the main features of the nonstationary analysis not be obscured by computational details, we shall restrict our discussions to a single-degree-of-freedom system and to two-dimensional incompressible flow as in the pioneering stationary analysis of Fung.[2] It is interesting to note, however, that such simplifications are commonly made at the preliminary design stage even for large flight vehicles. For the reader's convenient reference, some aspects of the random process theory required for the present analysis are reviewed in the Appendices.

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Index categories: Aircraft Gust Loading and Wind Shear; Aircraft Vibration; Structural Dynamic Analysis.

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‡ Theoretically, the turbulence velocity can not be exactly Gaussian distributed.[5]

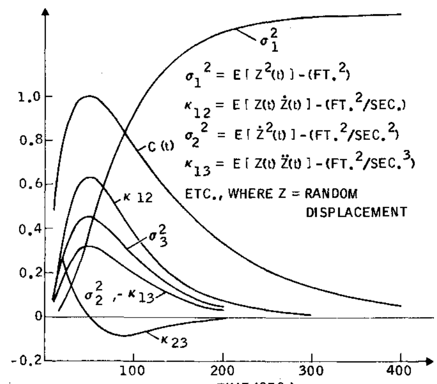

Fig. 1 Variation of nonstationary response statistics with time. Computations based on quasi-steady aerodynamic theory and the following physical data: $L = 500$ ft, $U = 200$ fps, $b = 3.0$ ft, $\lambda = 59.5$, $\sigma = 1.0$ fps, $c(t) = e^{-\alpha t} - e^{-\beta t}$, $\alpha = 0.009$, $\beta = 0.04$, $d = 1.11$.

Uniformly Modulated Gust Velocity

Consider a flight vehicle penetrating a turbulence field which is random in space and time. We assume that the turbulence field does not change appreciably during the time $(2b/U)$ required for an air particle to pass the lifting surface of the vehicle where U is the forward velocity of the vehicle and b is the semichord length of the lifting surface. Then, with respect to a coordinate system fixed to the vehicle, the gust velocity as sensed by the flight vehicle is a function of $x - Ut$, y, and z, i.e., $W = W(x - Ut,y,z)$ where the x axis is pointing in the opposite direction of the forward velocity. In the present study, however, we shall be concerned only with the vertical rigid body motion of the flight vehicle. This, in effect, reduces the airplane to a mass point and the system to one of single degree of freedom. Then the coordinate values are restricted to $x = y = z = 0$. This special case will be implied in the following and we shall regard the gust velocity to be merely a function of time.

Although the gust field is considered to be nonstationary in the present study we assume that it may be written as

$$W(t) = c(t)G(t) \tag{1}$$

where $c(t)$ is a deterministic function of time and $G(t)$ is a weakly stationary random process. A nonstationary random process expressible in a product form as in Eq. (1) is called a uniformly modulated random process and is potentially very useful in practical applications.§ By its very nature the uniformly modulated random process given by Eq. (1) is easily adaptable to the evolutionary spectral analysis. (See Appendix A). We therefore write

$$W(t) = \int_{-\infty}^{\infty} c(t)e^{i\omega t}d\tilde{W}(\omega) \tag{2}$$

That is, we choose $A(t,\omega) = c(t)$ in Eq. (A1). This particular choice also enables us to write

$$\Phi_{\tilde{W}\tilde{W}}(\omega) = \Phi_{GG}(\omega) \tag{3}$$

We now further assume that $G(t)$ is a Gaussian process with a zero mean and that its spectral density is the Dryden spectrum

$$\Phi_{GG}(\omega) = (\sigma^2 L/2\pi U)[1 + 3(\omega L/U)^2]/[1 + (\omega L/U)^2]^2 \tag{4}$$

§ The idea of uniformly modulated process was first used by Shinozuka.[7,8] Recently, Gaonkar and Hohenemser[4] also used it in the analysis of rotor blade vibrations.

where L is the scale of turbulence and σ^2 is the variance of $G(t)$. It can be shown, by computing its characteristic functional,[10,11] that $W(t)$ is also a Gaussian process, and so is the response of a linear system under the excitation of $W(t)$.

To proceed further it is necessary to select a suitable modulating function $c(t)$. For this purpose, we choose a versatile form

$$c(t) = \begin{cases} c[\exp(-\alpha t) - \exp(-\beta t)], & t > 0 \\ 0, & t < 0 \end{cases} \tag{5}$$

where $\beta > \sigma$ and $c > 0$. The normalization constant c is determined from the condition

$$\sup_t |c(t)| = 1$$

where sup denotes the least upper bound. It is of interest to remark that by a suitable choice of α and β values, $c(t)$ can be made to resemble rather well some of the "gust profiles" traditionally used in deterministic discrete gust analysis.[12]

Some Numerical Results

To demonstrate the feasibility of the nonstationary gust analysis proposed herein numerical results have been obtained for the statistics of the response using both quasi-steady and unsteady aerodynamic theories. The impulse-response function for the vertical displacement of the flight vehicle based on a quasi-steady aerodynamic theory is given by[13]

$$h(t) = [1 - \exp(-dt)]\mathbf{1}(t) \tag{6}$$

where $\mathbf{1}(t)$ is Heaviside's unit step function and d is a constant which is related to the mass parameter λ by $d = 2U/[b(2\lambda + 1)]$. The mass parameter $\lambda = m/(\pi\rho Ab)$ is computed from the mass of the flight vehicle m, the air density ρ, the wing area A, and the semichord length b. Substituting Eqs. (5)¶ and (6) into Eq. (A6) we obtain the real part u and the imaginary part v of $M(t,\omega)$ as follows:

$$u(t,\omega) = f_e(0,\alpha) - f_e(0,\beta) - f_e(d,\alpha) + f_e(d,\beta) \tag{7a}$$

$$v(t,\omega) = f_0(0,\alpha) - f_0(0,\beta) - f_0(d,\alpha) + f_0(d,\beta) \tag{7b}$$

where

$$f_e(d,x) = \{(x - d)[\exp(-dt)\cos\omega t - \exp(-xt)] + \omega\exp(-dt)\sin\omega t\}/[(x - d)^2 + \omega^2] \tag{8a}$$

$$f_0(d,x) = \{\omega[\exp(-dt)\cos\omega t - \exp(-xt)] - (x - d)\exp(-dt)\sin\omega t\}/[(x - d)^2 + \omega^2] \tag{8b}$$

These can then be substituted into the expressions for variances and covariances in (B12) and the integrals can be evaluated** by use of the method of residues. Figure 1 shows the numerical results of such computations for the particular set of physical parameters given in the caption. Since these statistics are of different units, the individual curves in Fig. 1 only illustrate the relative magnitude variations for each. The nonstationary characteristic of the response is clearly demonstrated in that the variances are functions of time and the covariance κ_{12} is generally nonzero. As the modulating function tends to zero all the statistics of the response vanish except that σ_1^2 approaches to a nonzero constant. This is to be expected since there is an uncertainty in the aircraft altitude after it has passed through a gust field.

Having determined the variances and covariances for displacement $Z(t)$ and its derivatives $d/dt\,Z(t)$ and $d^2/dt^2\,Z(t)$ we can compute the probability density $p_{\Xi}(\xi,t)$ for the displacement peak at time t on the condition that a peak does occur at this instant using Eq. (B11). In Fig. 2a we have plotted this probability density at $t = 30$ sec. Also shown in this

¶ Recall that $c(t) \equiv A(t,\omega)$ here.

** The lengthy algebraic details may be found in Ref. 11.

figure are Gaussian and Rayleigh probability densities, $p_1(\xi)$ and $p_2(\xi)$, using the same parameter σ_1, i.e.

$$p_1(\xi) = (2\pi)^{-1/2}\sigma_1^{-1}\exp[-\xi^2/(2\sigma_1^2)],\ -\infty < \xi < \infty$$

$$p_2(\xi) = \sigma_1^{-2}\xi\exp[-\xi^2/(2\sigma_1^2)],\ 0 \le \xi < \infty$$

It is seen that the peak magnitude distribution is much closer to the Gaussian law than it is to the Rayleigh law, suggesting that the displacement response may be a wide-band random process.[14] Indeed, this has been confirmed by the fact that the computed average zero up-crossing rate (not shown) is much smaller than the average total number of peaks per unit time. As illustrated in Fig. 2b the peak distribution becomes indistinguishable from the Gaussian law at $t = 150$ sec.

So far we have taken the vertical rigid body displacement of the flight vehicle as the required system response, and the numerical results reported above have shed considerable light on the merit of using a nonstationary gust model for such an analysis. However, since it is the vertical acceleration that can be measured experimentally, a more meaningful response quantity may be the acceleration. The mean square value of the acceleration has been plotted in Fig. 1 (same as the variance σ_3^2 in the figure) which can be obtained from the equation (B12f) by substituting for u and v the expressions in Eq. (7). More simply we can begin from the following impulse-response function for the acceleration:

$$h_a(t) = d^2/dt^2h(t) = d\delta(t) - d^2\exp(-dt)1(t) \quad (9)$$

If this impulse-response function is used in Eq. (A6) we obtain the corresponding

$$M_a(t,\omega) = \int_0^t h_a(\tau)A(t-\tau,\omega)\exp(-i\omega\tau)d\tau = u_a(t,\omega) + iv_a(t,\omega) \quad (10)$$

Substituting $c(t)$ from Eq. (5) for $A(t,\omega)$, we obtain

$$u_a(t,\omega) = d[\exp(-\alpha t) - \exp(-\beta t)] + d^2[(d-\alpha)\exp(-dt)\cos\omega t - (d-\alpha)\times \exp(-\alpha t) - \omega\exp(-dt)\sin\omega t]/[(d-\alpha)^2 + \omega^2] - d^2[(d-\beta)\exp(-dt)\cos\omega t - (d-\beta)\exp(-\beta t) - \omega\exp(-dt)\sin\omega t]/[(d-\beta)^2+\omega^2] \quad (11)$$

$$v_a(t,\omega) = d^2[\omega\exp(-\alpha t) - (d-\alpha)\exp(-dt)\sin\omega t - \exp(-dt)\cos\omega t]/[(d-\alpha)^2+\omega^2] - d^2[\omega\exp(-\beta t) - (d-\beta)\exp(-dt)\sin\omega t - \exp(-dt)\omega\cos\omega t]/[(d-\beta)^2+\omega^2] \quad (12)$$

When u_a and v_a replace u and v in the expressions (B12) we obtain variances σ_{j+2}^2 instead of σ_j^2 and covariances $\kappa_{j+2,k+2}$ instead of κ_{jk}.

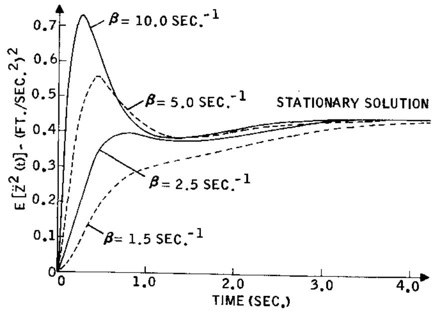

Fig. 3 Mean square plunging acceleration due to a nonstationary gust. Computations based on quasi-steady aerodynamic theory and the following physical data: L = 500 ft, U = 200 fps, b = 3.0 ft, λ = 59.5, σ = 1.0 fps, $c(t) = 1 - e^{-\beta t}$, d = 1.11 sec^{-1}.

Figure 3 shows the results for $\sigma_3^2 = E[\ddot{Z}^2(t)]$ corresponding to the case $\alpha = 0$; that is, to a gust modulating function $c(t) = 1 - \exp(-\beta t)$. We see that as β increases in magnitude (corresponding to a steeper modulating gradient) the transient response becomes more pronounced. This general trend has been shown to be present in the deterministic (discrete gust) analysis.[13] It can be shown that σ_3^2 approaches the correct stationary result (for all values of β) as $c(t)$ tends to unity. It is interesting to note that the transient overload would not be detected in a stationary analysis.

When $\alpha \neq 0$ the integrations become more laborious. However, if the constant d is much larger than both α and β, a useful approximation can be devised by replacing both $d - \alpha$ and $d - \beta$ by $d - (\alpha + \beta)/2$. Then it can be shown that

$$\sigma_3^2(t) \cong \frac{\sigma^2 d^2}{2}\,\frac{c^2(t)[2 - (\Delta L/U) - 4(\Delta L/U)^2 + 3(\Delta L/U)^3]}{[1 - (\Delta L/U)^2]^2} \quad (13)$$

where $\Delta = d - (\alpha + \beta)/2$ and we have also used $d^2 \cong d\Delta$. The approximate solution given by Eq. (13) is shown in Fig. 4 along with the "exact" solution [neglecting no terms and using Eqs. (11) and (12)]. We see from Fig. 4 that the approximation yields sufficiently accurate results for the case shown. The restriction imposed on the approximate formulation is simply that the gust modulating function must be slowly varying with time.

It is generally recognized that the quasi-steady aerodynamic theory is adequate when the scale of turbulence L is much larger than the semichord length b. Some preliminary results, however, have also been obtained by use of a mixed un-

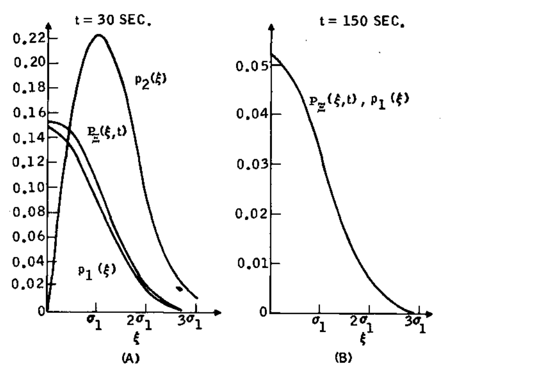

Fig. 2 Probability density of peak magnitude conditional on the occurrence of a peak at t = 30 sec and t = 150 sec. Computations based on quasi-steady aerodynamic theory and physical data given in Fig. 1.

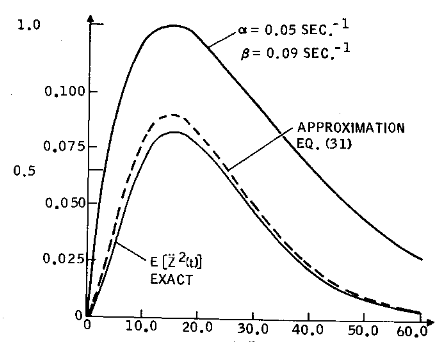

Fig. 4 Mean square plunging acceleration due to another nonstationary gust. Computations based on quasi-steady aerodynamic theory and the following physical data: U = 125 fps, b = 4.0 ft, λ = 50.0, turbulence scale = 500 ft, d = 0.618, σ = 1.0 fps.

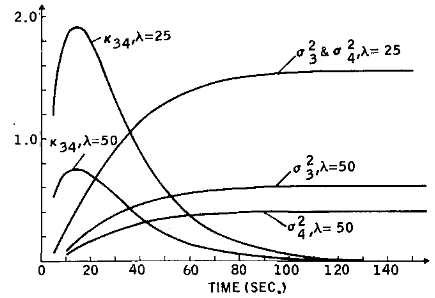

Fig. 5 Variances and covariances of $(d^2/dt^2)Z(t)$, and $(d^3/dt^3)Z(t)$. Computations based on a mixed quasi-steady and unsteady aerodynamic theory and the following physical data. $L = 200$ ft, $U = 200$ fps, $b = 3.5$ ft, $\sigma^2 = 1.0$ fps^2, $c(t) = 1 - \exp(-0.05\ t)$.

steady and quasi-steady approach which was used by Fung[2] in the stationary gust analysis. These results, to be briefly discussed below, do not include the beginning transient motion in the response.

For a passive system it is generally possible to establish an exponental bound for the impulse-response function $h(\tau)$; that is, there exists a constant γ such that $|h(\tau)| \leq \exp(-\gamma\tau)$. We shall now be concerned only with the response at $t \gg 1/\gamma$. We further assume that the modulating function $c(t)$ is a slowly varying function such that $|c(t - \tau) - c(t)| \ll |c(t)|$ for $|\tau|$ less than, say, $3/\gamma$. Under these conditions, Eq. (A6) can be simplified to

$$M(t,\omega) \approx \int_0^\infty h(\tau)c(t) \exp(-i\omega\tau) \approx c(t)H(\omega) \quad (14)$$

where $H(\omega)$ is the frequency-response function of the system. The response statistics obtained from Eq. (14) are then

$$\sigma_1{}^2 \approx c^2(t) \int_{-\infty}^{\infty} \Phi_{GG}(\omega)|H(\omega)|^2 d\omega \quad (15a)$$

$$\kappa_{23} \approx c(t)c'(t) \int_{-\infty}^{\infty} \Phi_{GG}(\omega)|H(\omega)|^2 d\omega \quad (15b)$$

$$\sigma_2{}^2 \approx c^2(t) \int_{-\infty}^{\infty} \omega^2\Phi_{GG}(\omega)|H(\omega)|^2 d\omega, \text{ etc.} \quad (15c)$$

The preceding expressions are intuitively plausible. In particular, the nonstationary variances are approximated by the stationary variances multiplied by the square of the modulating function. It is interesting to remark that Priestly[15,16] has suggested an approximation for slowly modulated processes which is essentially the same as the first expression in Eq. (15). Here we have given a more specific physical interpretation.

In Fung's work the frequency-response function for the acceleration is factored into two parts††:

$$H_a(\omega) = \eta_2(\omega)\eta_1(\omega) \quad (16)$$

where, $\eta_1(\omega)$ is the admittance of the lift to a sinusoidal gust and $\eta_2(\omega)$ is the frequency response of the acceleration to a sinusoidal lift.

For an incompressible flow

$$\eta_1(\omega) = 2^{1/2}\pi\rho b U\phi(k) \quad (17)$$

where $\phi(k)$ is the Sears function.[17] Fung used Liepmann's approximation[18]

$$|\phi(k)|^2 \approx [1 + 2\pi|k|]^{-1} \quad (18)$$

and a quasi-steady approximation for $\eta_2(\omega)$ as follows:

$$\eta_2(\omega) = 2^{1/2}k\{\pi\rho A b[k(1 + 2\lambda) - 2i]\}^{-1} \quad (19)$$

Therefore, his approach was not based on a strictly unsteady aerodynamic theory. On the other hand, in a purely quasi-steady approach, the Sears function in Eq. (17) would have been replaced by the constant unity. We remark that comparative calculations[11] have shown that this refinement is unnecessary when b/L is less than 0.01.

†† $H_a(\omega) = \omega^2 H(\omega)$.

Figure 5 shows the statistics $\sigma_3{}^2$, $\sigma_4{}^2$, and κ_{34} of $d^2/dt^2Z(t)$ and $d^3/dt^3Z(t)$ computed by use of Eq. (16) and for a modulating function $c(t) = 1 - \exp(-\beta t)$, $\beta = 0.05$ sec^{-1}. However, instead of Liepmann's approximation for the Sears function, we have used

$$\phi(k) \approx \phi_A(k) = [0.065/(0.13 + ik) + 0.5/(1 + ik)] \exp(1.10ik) \quad (20)$$

The values of this approximation are plotted in Fig. 6 along with the exact Sears function.‡‡ We see that $\phi_A(k)$ is reasonably accurate in the range of reduced frequency $k \leq 2$. Since most of the turbulence energy in the Dryden spectrum is contained in the low-frequency range this approximation appears to be satisfactory. In Fig. 7 we compare $|\phi(k)|^2$, $|\phi_A(k)|^2$ and Liepmann's approximation $(1 + 2\pi k)^{-1}$. We see that $|\phi_A(k)|^2$ is slightly conservative for small k, whereas, Liepmann's approximation is slightly nonconservative. However, the main reason for using $\phi_A(k)$ is the ability to compute $\sigma_4{}^2$. If Liepmann's approximation were used $\sigma_4{}^2$ would be unbounded. This brings out a subtle point in the choice of approximate aerodynamic admittances. $\phi_A(k)$ would be also unsuitable if we wish to compute, say, $\sigma_5{}^2$.

With the knowledge of $\sigma_3{}^2$, $\sigma_4{}^2$ and κ_{34}. We can compute the threshold crossing statistics of the acceleration $d^2/dt^2 Z(t)$. This is shown in Fig. 8 for three threshold levels 1.0, 2.0 and 3.0 ft/sec^2.

Concluding Remarks

As was demonstrated in the above sample computations the modeling of atmospheric turbulence by a nonstationary random process is not a great deal more complicated mathematically than the usual stationary analysis. It offers a number of advantages over the usual method, the most obvious one is its much greater versatility to fit different geographical peculiarities and flight missions. The use of a uniformly modulated random process is especially attractive since it combines the prominent features of the deterministic analysis with discrete gust profile and the probabilistic analysis with stationary input. Some government purchasing agencies usually specify that analyses must be carried out for both deterministic and random models, perhaps, for fear of neglecting transient overloads. Figure 3 dramatically illustrates the merit of the present analysis to account for such overloads. The assumption of nonstationarity in the input process is considered more direct; it is also on a sounder mathematical basis than a proposed modification to the usual stationary model by taking the variance of the gust velocity

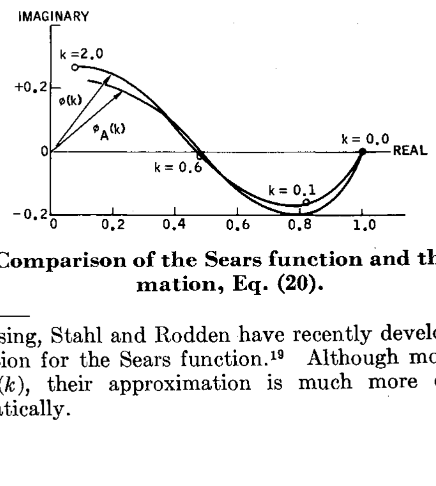

Fig. 6 Comparison of the Sears function and the approximation, Eq. (20).

‡‡ Giesing, Stahl and Rodden have recently developed an approximation for the Sears function.[19] Although more accurate than $\phi_A(k)$, their approximation is much more complicated mathematically.

as a random variable[6] since any statistical average is, by definition, deterministic.

In Eq. (1) the modulating envelope $c(t)$ is assumed to be deterministic; however, the analysis can be extended to the case where $c(t)$ is a random process but statistically independent of $G(t)$. The extension can be accomplished simply by substituting $E[c(t)]$ and $E[c(t_1)c(t_2)]$ for $c(t)$ and $c(t_1)c(t_2)$, respectively, where $E[\ ]$ denotes the ensemble average of the bracketed random quantity.

The flight speeds chosen in the numerical examples are well within the range where incompressible flow is an acceptable assumption, in line with the simple nature of an exploratory analysis. For higher flight speeds the compressibility effect must be taken into consideration. We note, however, that the results presented have been calculated using parameters representative of certain realistic flight conditions. In particular, the conditions simulate those experienced by small aircraft at low-altitude and speed.

We have used herein the evolutionary spectral densities to characterize nonstationary input and output.[15,16] Still other tools are available.[23] Among these the generalized spectra and the instantaneous spectra[24] appear to be suitable for application to linear systems, and exploratory studies have also been made[11] on such possibilities.

Appendix A: Evolutionary Spectral Relationship Between Input and Output of a Linear System

The concept of evolutionary spectrum was introduced by Priesley[15,16] and was used by Hammond[25] and Shinozuka[26] among others in the study of response of linear systems to random excitations. Let the input random process be expressed as a stochastic Stieltjes integral

$$W(t) = \int_{-\infty}^{\infty} A(t,\omega)e^{i\omega t}d\tilde{W}(\omega) \tag{A1}$$

where $A(t,\omega)$ is a suitable deterministic function of t and ω and $\tilde{W}(\omega)$ is an orthogonal random process with the property:

$$\begin{aligned} E[d\tilde{W}(\omega_1)d\tilde{W}^*(\omega_2)] &= 0, \quad \omega_1 \neq \omega_2 \\ E[|d\tilde{W}(\omega)|^2] &= \Phi_{\tilde{W}\tilde{W}}(\omega)d\omega \end{aligned} \tag{A2}$$

We shall assume that $W(t)$ is defined on the positive t domain. Then $A(t,\omega) = 0$ for $t < 0$.

The representation Eq. (A1) is not unique since a change in the choice of $A(t,\omega)$ can be accommodated by a corresponding change in $\tilde{W}(\omega)$. The family of suitable $A(t,\omega)$ $\exp(i\omega t)$ functions for the previous representation is called the family of oscillatory functions.[15,16] It is convenient, however, to normalize $A(t,\omega)$ such that

$$\sup_{t,\omega}|A(t,\omega)| = 1$$

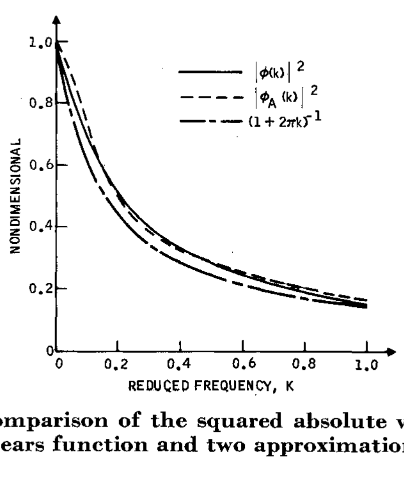

Fig. 7 Comparison of the squared absolute value of the Sears function and two approximations.

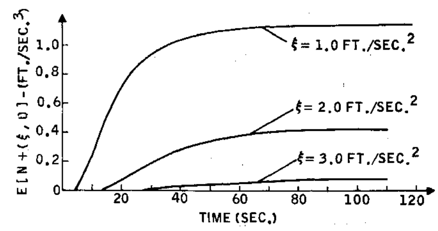

Fig. 8 Ensemble average of threshold-crossing rate of the plunging acceleration. Computations based on a mixed quasi-steady and unsteady aerodynamic theory and the following physical data: L = 200 ft, U = 200 fps, b = 3.5 ft, λ = 25, σ = 1.0 fps.

Then in the special case where $W(t)$ is a weakly stationary process, $A(t,\omega)$ reduces to the constant value unity and $\Phi_{\tilde{W}\tilde{W}}(\omega)$ coincides with the spectral density of $W(t)$. In the general case $\Phi_{\tilde{W}\tilde{W}}(\omega)$ is the spectral density of some suitable weakly stationary random process.

Since $W(t)$ is a real-valued random process, we have both

$$W(t_1) = \int_{-\infty}^{\infty} A(t,\omega_1) \exp(i\omega_1 t_1)d\tilde{W}(\omega_1)$$

and

$$W(t_2) = \int_{-\infty}^{\infty} A^*(t_2,\omega_2) \exp(-i\omega_2 t_2)d\tilde{W}^*(\omega_2)$$

The autocorrelation function of $W(t)$ is obtained by taking the ensemble average of the product of $W(t_1)$ and $W(t_2)$ and using the orthogonality property of $\tilde{W}(\omega)$, the two equations in Eq. (A2),

$$\phi_{WW}(t_1,t_2) = E[W(t_1)W(t_2)] = \int_{-\infty}^{\infty} A(t_1,\omega)A^*(t_2,\omega) \exp[i\omega(t_1 - t_2)]\Phi_{\tilde{W}\tilde{W}}(\omega)d\omega \tag{A3}$$

In particular, we see that

$$\begin{aligned} E[W^2(t)] &= \int_{-\infty}^{\infty} f_w(t,\omega)d\omega, \\ f_w(t,\omega) &= |A(t,\omega)|^2\Phi_{\tilde{W}\tilde{W}}(\omega) \end{aligned} \tag{A4}$$

where $f_W(t,\omega)$ is called the evolutionary spectral density of $W(t)$.

Let us digress to comment on the physical meaning of Eqs. (A1) and (A4). We recall that in the special case of a weakly stationary random process, $A(t,\omega)$ reduces to the constant unity. Thus, for this special case, Eq. (A1) is clearly the Fourier decomposition (physically the frequency decomposition) of $W(t)$ and for different ω, $d\tilde{W}(\omega)$ are the orthogonal coefficients of the decomposition. As seen in Eq. (A2), the ensemble average of the squared magnitude of each such coefficient gives the average energy content in $(w, w + dw)$.

We wish to retain a similar decomposition when $W(t)$ is nonstationary. This is accomplished by replacing $A(t,\omega)$ $d\tilde{W}(\omega)$ for the coefficients. These coefficients vary with time but they remain to be orthogonal. The average energy content in each component is, again, obtained from averaging the squared magnitude of the corresponding coefficient. As seen from (A4), the evolutionary spectrum $f(t,\omega)$ represents the distribution of the total average energy over the ω-domain for every t. It is not surprising to find the average energy distribution time-dependent since, by definition, the probabilistic structure of a nonstationary random process is changing with time.[16]

Now, consider a linear time-invariant system characterized by an impulse-response function $h(t)$ or, equivalently, by a frequency-response function

$$H(\omega) = \int_0^{\infty} h(t) \exp(-i\omega t)dt$$

Under the excitation $W(t)$ given in Eq. (A1) the response process $z(t)$ may be expressed also as a stochastic Stieltjes integral

$$Z(t) = \int_{-\infty}^{\infty} M(t,\omega) \exp(i\omega t)\, d\tilde{W}(\omega) \tag{A5}$$

with

$$M(t,\omega) = \int_0^t h(\tau)A(t-\tau,\omega) \exp(-i\omega\tau)d\tau \tag{A6}$$

where the assumption has been made that the system was initially at rest. The autocorrelation of the response may be obtained readily

$$\phi_{ZZ}(t_1,t_2) = E[Z(t_1)Z(t_2)] = \int_{\infty}^{\infty} M(t_1,\omega)M^*(t_2,\omega) \exp[i\omega(t_1-t_2)]\Phi_{\tilde{W}\tilde{W}}(\omega)d\omega \tag{A7}$$

The mean square value of the response follows from letting $t_1 = t_2 = t$

$$E[Z^2(t)] = \int_{-\infty}^{\infty} |M(t,\omega)|^2\Phi_{\tilde{W}\tilde{W}}(\omega)d\omega \tag{A8}$$

It may be noted that both $A(t,\omega)$ and $M(t,\omega)$ are hermitian with respect to ω. Write $M(t,\omega) = u(t,\omega) + iv(t,\omega)$. We see that u and v are even and odd functions of ω, respectively. In the special case of a weakly stationary excitation (the case $A \equiv 1$), $M(t,\omega)$ tends to the frequency-response function $H(\omega)$ as t becomes large.

For what follows we shall assume that the ensemble average of the excitation $W(t)$ is zero; then the ensemble average of the response $Z(t)$ is also zero. In this case Eqs. (A7) and (A8) are also the expressions for the covariance and the variance of the response, respectively.

Appendix B: Threshold-Crossing and Peak Distribution of the Response

The statistics of threshold crossing and peak distribution of the response are required for the estimate of reliability of structural components[10] or electronic devices. These statistics are also indicative of ride roughness; therefore, their bounds may be specified for different types of flight missions from the standpoint of human tolerance. The ensemble average of the random rate at which the response $Z(t)$ crosses the threshold level ξ at a positive slope (called up-crossings) is given by[27]

$$E[N_+(\xi,t)] = \int_0^{\infty} \dot{z}p_{12}(\xi,\dot{z},t)d\dot{z} \tag{B1}$$

where p_{12} is the joint probability density of $Z(t)$ and $d/dt\, Z(t)$. When $Z(t)$ is a Gaussian random process

$$p_{12}(z,\dot{z},t) = (2\pi\sigma_1\sigma_2)^{-1}(1-\rho_{12}^2)^{-1/2} \exp\{[-(z/\sigma_1)^2 + 2\rho_{12}(z/\sigma_1)(\dot{z}/\sigma_2) - (\dot{z}/\sigma_2)^2]/[2(1-\rho_{12}^2)]\} \tag{B2}$$

where σ_1^2 = variance of $Z(t)$, as given by Eq. (A8), σ_2^2 = variance of $d/dt\, Z(t)$, and ρ_{12} = correlation coefficient of $Z(t)$ and $d/dt\, Z(t)$. Substituting Eq. (B2) into Eq. (B1) and integrating, we obtain

$$E[N_+(\xi,t)] = \sigma_2(2\pi\sigma_1)^{-1}(1-\rho_{12}^2)^{1/2} \times \exp(-\eta^2/2)\ [\exp(-\mu^2) + \pi^{1/2}\mu(1+\text{erf}\mu)] \tag{B3}$$§§

where erf is the error function, and

$$\eta = \xi/\sigma_1$$

$$\mu = \eta\rho_{12}[2(1-\rho_{12}^2)]^{-1/2}$$

In particular, the average zero-crossing rate (i.e. the case $\xi = 0$) is

$$E[N_+(0,t)] = (1-\rho_{12}^2)^{1/2}\sigma_2/(2\pi\sigma_1) \tag{B4}$$

The average number of peaks per unit time in the response above a threshold level ξ is given by[27]

$$E[M(\xi,t)] = -\int_{\xi}^{0} dz \int_{-\infty}^{0} \ddot{z}p_{123}(z,0,\ddot{z},t)d\ddot{z} \tag{B5}$$

where p_{123} is the joint probability density of $Z(t)$, $d/dt\, Z(t)$ and $d^2/dt^2\, Z(t)$. The average number of peaks per unit time regardless of magnitude is obtained from Eq. (B5) by letting $\xi = -\infty$; i.e.,

$$E[M_T(t) = E[M(-\infty,t)] \tag{B6}$$

For a smooth random process, the probability that two or more peaks occur in an infinitesimal time interval $(t,t+dt)$ is negligible compared with the probability of the occurrence of one peak in that interval. Then $E[M_T(t)]dt$ = Prob {one peak occurs in $(t+dt)$} and $E[M(\xi,t)]dt$ = Prob {one peak above ξ occurs in $(t,t+dt)$} and the expression

$$F_{\Xi}(\xi,t) = 1 - E[M(\xi,t)]/E(M_T(t)] \tag{B7}$$

gives the probability for the peak magnitude at time t to be equal to or less than ξ on the condition that there is indeed a peak at this time instant. The function $F_{\Xi}(\xi,t)$ is, therefore, the conditional probability distribution function of the peak magnitude. The conditional probability density of peak magnitude is obtained by differentiating Eq. (B7)

$$P_{\Xi}(\xi,t) = -d/d\xi E[M(\xi,t)]/E[M_T(t)] = -1/E[M_T(t)] \int_{-\infty}^{0} \ddot{z}p_{123}(\xi,0,\ddot{z},t)d\ddot{z} \tag{B8}$$

Again, when $Z(t)$ is a Gaussian random process,

$$p_{123}(y_1,y_2,y_3) = (2\pi)^{-3/2}|S|^{-1/2} \times \exp\left\{-\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\alpha_{jk}y_jy_k\right\} \tag{B9}$$

where y_1, y_2, y_3 have replaced z, $\dot{z}$, $\ddot{z}$, where $|S|$ is the determinant of the matrix of variances and covariances for $Z(t)$, $d/dtZ(t)$ and $d^2/dt^2Z(t)$; i.e.,

$$|S| = \begin{vmatrix} \sigma_1^2 & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \sigma_2^2 & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \sigma_3^2 \end{vmatrix} \tag{B10}$$

and where α_{jk} are the (j,k) element of $\mathbf{S}^{-1}$. Upon substituting Eq. (B9) into Eq. (B8) and carrying out the integration, we obtain

$$p_{\Xi}(\xi,t) = (2\pi)^{-1/2}[\alpha_{11}^{1/2} - \alpha_{13}^2/(\alpha_{11}^{1/2}\alpha_{33})] \times \{\exp(-\alpha_{11}\xi^2/2) + \xi(\pi/2)^{1/2}(\alpha_{13}/\alpha_{33}^{1/2}) \times \exp[-(\alpha_{11} - \alpha_{13}^2/\alpha_{33})\xi^2/2] \times [1 + \text{erf}(2^{-1/2}\alpha_{33}^{-1/2}\alpha_{13}\xi)]\} \tag{B11}$$

The application of Eqs. (B3), (B4), and (B11) requires knowledge of the elements of the determinant in Eq. (B10). As was noted previously, the variance σ_1^2 of $Z(t)$ is given by Eq. (A8) which involves the function $M(t,\omega) = u(t,\omega) + iv(t,\omega)$. The other variances and covariances may be obtained by differenting Eq. (A7) with respect to t_1 and/or t_2 and then letting $t_1 = t_2 = t$.

The results are summarized as follows:

$$\sigma_1^2 = \int_{-\infty}^{\infty} \Phi_{\tilde{W}\tilde{W}}(\omega)(u^2+v^2)d\omega \tag{B12a}$$

$$\kappa_{12} = \frac{1}{2}\frac{\partial}{\partial t}\sigma_1^2 \tag{B12b}$$

§§ This expression has been obtained previously by Roberts.[28]

$$\sigma_2^2 = \int_{-\infty}^{\infty} \Phi_{\tilde{W}\tilde{W}}(\omega)\left[\omega^2(u^2 + v^2) + 2\omega\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right]d\omega \quad \text{(B12c)}$$

$$\kappa_{13} = \frac{\partial}{\partial t}\kappa_{12} - \sigma_2^2 \quad \text{(B12d)}$$

$$\kappa_{23} = \frac{1}{2}\frac{\partial}{\partial t}\sigma_2^2 \quad \text{(B12e)}$$

$$\sigma_3^2 = \int_{-\infty}^{\infty} \Phi_{WW}(\omega)\left\{\left[\omega^4(u^2 + v^2) + 4\omega^2\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right] + \left(\frac{\partial^2 u}{\partial t^2}\right)^2 + \left(\frac{\partial^2 v}{\partial t^2}\right)^2 + 4\omega\left(\frac{\partial u}{\partial t}\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t}\frac{\partial^2 u}{\partial t^2}\right) - 2\omega^2\left(u\frac{\partial^2 u}{\partial t^2} + v\frac{\partial^2 v}{\partial t^2}\right) + 4\omega^3\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\right]\right\}d\omega \quad \text{(B12f)}$$

It is interesting to note that in the case of weakly stationary response, u and v are independent of t, and the expressions in (B12) reduce to

$$\sigma_1^2 = \int_{-\infty}^{\infty} \Phi_{\tilde{W}\tilde{W}}(\omega)(u^2 + v^2)d\omega$$

$$\sigma_2^2 = \int_{-\infty}^{\infty} \Phi_{\tilde{W}\tilde{W}}(\omega)(u^2 + v^2)\omega^2 d\omega \quad \text{(B12a)}$$

$$\sigma_3^2 = \int_{-\infty}^{\infty} \Phi_{\tilde{W}\tilde{W}}(\omega)(u^2 + v^2)\omega^4 d\omega$$

$$\kappa_{12} = \kappa_{23} = 0, \quad \kappa_{13} = -\sigma_2^2$$

When these special values are substituted into Eqs. (B3) and (B11) we obtained the respective known stationary expressions for example, Ref. 11:

$$E[N_+(\xi,t)] = \sigma_2/(2\pi\sigma_1)\exp(-\eta^2/2) \quad \text{(B3a)}$$

$$p_{\Xi}(\xi,t) = (2\pi)^{-1/2}(\sigma_1^2\sigma_2\sigma_3)^{-1} \times \{|S|^{1/2}\exp[-(\sigma_1\sigma_2\sigma_3\eta)^2/2|S|] + \sigma_2^3\eta(\pi/2)^{1/2} \cdot [1 + \text{erf}(\sigma_2^6\eta^2/2|S|)^{1/2}]\exp(-\eta^2/2)\} \quad \text{(B11a)}$$

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